A GENERALIZED DESIGN METHOD FOR DIRECTIVITY PATTERNS OF SPHERICAL MICROPHONE ARRAYS

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ABSTRACT

Spherical microphone arrays provide a flexible solution to obtaining higher-order directivity patterns, which are useful in audio recording and reproduction. A general systematic approach to the design of directivity patterns for spherical microphone arrays is introduced in this paper. The directivity patterns are obtained by optimizing a cost function which is a convex combination of a front-back energy ratio and a smoothness term. Most of the standard directivity patterns - i.e. omnidirectional, cardioid, subcardioid, hypercardioid and supercardioid - are particular solutions of this optimization problem with specific values of two free parameters: the angle of the frontal sector, and the convex combination factor. By varying these two parameters, more general solutions of practical use are obtained.

Index Terms— Microphones, directivity pattern, beamforming, spherical microphone array.

1. INTRODUCTION

Directional microphones have been a subject of research since the rise of commercial broadcasting in the early 1920s. Their development was based on the need to emphasize the voice of news presenters by suppressing surrounding noise sources. One of the earliest directional microphones is the *cardioid* microphone. The amplitude response to a plane wave incident from the direction θ - commonly referred to as *directivity pattern* - for the cardioid microphone is $\Gamma(\theta) = 0.5 + 0.5\cos\theta$. Most of the directional microphones used in the recording industry today are members of the *first-order* cardioid family, i.e. microphones whose directivity patterns are of the kind $\Gamma(\theta) = (1-\gamma) + \gamma\cos\theta$, with $\gamma \in [0, 1]$.

The majority of the microphones available in the market today serve specific purposes, and have fixed directivity patterns. For $\gamma = 0.5$ the resulting directivity is cardioid, whereas *subcardioid* has $\gamma = 0.3$ [1]. Other patterns like *supercardioid* and *hypercardioid* are designed to satisfy specific criteria. Supercardioid is defined as the pattern that exhibits the maximum *front-back ratio* ($\gamma = 0.63$), whereas hypercardioid is the microphone with the maximum *directivity index* ($\gamma = 0.75$) [1].

Due to the high cost and lack of flexibility of having a different microphone for each application, several techniques to obtain variable-pattern microphones have been developed since the mid 1930s [1]. Usually a slider serves as the interface to set the value of γ for variable-pattern microphones. Sound engineers and technicians set the value so as to obtain one of the above mentioned patterns, or hybrids between them, according to their taste and the requirements of the recording scenario.

Recently, there has been significant interest in spherical microphone arrays [2, 3, 4, 5]. Meyer and Elko [3] proposed an array composed of a number of pressure microphones appropriately positioned on a rigid sphere. This microphone array can be used in several ways, including sound-field recordings, sound-field analysis, and beamforming. Higher-order directivity patterns of the kind $\Gamma(\theta) = \sum_{n=0}^{N} a_n \cos^n(\theta)$ can be obtained by such microphone, greatly enlarging the design space of available patterns.

The coefficients $a_0, ..., a_N$ of higher-order versions of standard directivity patterns are already known [6]. Although these standard patterns meet the needs of sound engineers up to a certain degree, directivity patterns corresponding to less specific criteria are also of practical use. As the order of the microphone increases, obtaining such directivity patterns by individually adjusting the N coefficients becomes difficult, if not impossible. A design framework to overcome this limitation is presented in this paper.

This paper is organized as follows. In Sec. 2 a very brief overview of the theory of spherical microphone arrays is given, and the standard directivity patterns are characterized. The generalized design method for microphone directivity patterns is presented in Sec. 3. Conclusions are drawn in Sec. 4.

2. BACKGROUND

The pressure field due to a plane wave with wavenumber k, incident from the direction (θ, ϕ), as observed on a sphere Ω_s centered in the origin of the reference system, can be expanded as [7, 4, 8]:

$$p(\mathbf{r}_s, \mathbf{k}) = 4\pi \sum_{n=0}^{\infty} i^n f_n(kr_s) \sum_{m=-n}^{n} Y_n^m(\theta, \phi) Y_n^m(\theta_s, \phi_s)^*, \quad (1)$$

where $i = \sqrt{-1}$ is the complex unit, $(\cdot)^*$ denotes the complex conjugate, Y_n^m is the spherical harmonic of order n and degree m, $\mathbf{r}_s = [r_s, \theta_s, \phi_s]$ identifies the point on the sphere of radius r_s, and

$$f_n(x) = \begin{cases} j_n(x) & \Omega_s \text{ open sphere} \\ j_n(x) - \frac{j_n'(x)}{h_n'(x)} h_n(x) & \Omega_s \text{ rigid sphere,} \end{cases} \quad (2)$$

where j_n is the spherical Bessel function of order n, and h_n are the spherical Hankel functions of first kind. The spherical harmonics exhibit the property of being orthonormal:

$$\int_{\Omega_s} Y_n^m(\theta_s, \phi_s) Y_{n'}^{m'}(\theta_s, \phi_s)^* \, d\Omega_s = \delta_{n-n'} \delta_{m-m'}. \quad (3)$$

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Let us now assume that the surface of the sphere Ω_s is populated by infinitely many infinitesimal pressure microphones. If the pressure recorded at each point is weighted by:

$$W_{n'}^{m'}(\theta_s, \phi_s, kr_s) = \frac{Y_{n'}^{m'}(\theta_s, \phi_s)}{4\pi i^{n'} f_{n'}(kr_s)}, \quad (4)$$

the total output is:

$$\int_{\Omega_s} p(\mathbf{r}_s, \mathbf{k}) W_{n'}^{m'}(\theta_s, \phi_s, kr_s)\, d\Omega_s = Y_{n'}^{m'}(\theta, \phi), \quad (5)$$

where the property (3) has been used [7]. This shows that the spatial response to a plane-wave incident from the direction (θ, ϕ), commonly referred to as *directivity pattern*, is exactly $Y_{n'}^{m'}(\theta, \phi)$. By using different values of n' and m' in (4), all spherical harmonic components can be obtained, and therefore it is possible to synthesize any square integrable directivity function [8] as a weighted infinite sum of spherical harmonics.

The conclusion of eq. (5) is valid under the assumption that the pressure at each arbitrary point is known, which is not feasible. However, if the surface is sampled using pressure microphones at specific points [7, 4, 5], the discrete version of property (3) holds for spherical harmonics up to a certain order, N. This upper bound N depends on the number of pressure microphones, and the chosen spatial sampling pattern.

In the context of this paper, only spherical harmonics of degree $m = 0$ are used. In other words, only axisymmetrical (ϕ-independent) directivity patterns are considered in this paper. In this way directivity patterns of the kind:

$$\Gamma(\theta) = \sum_{n=0}^{N} b_n Y_n^0(\theta, \phi) = \sum_{n=0}^{N} b_n \sqrt{\frac{2n+1}{4\pi}} P_n(\cos\theta) \quad (6)$$

are obtained, where $P_n(x)$ is the Legendre function of the first kind. Since $P_n(x)$ is a polynomial of degree n, the directivity function can be expressed as:

$$\Gamma_{\mathbf{a}}(\theta) = a_0 + a_1 \cos\theta + a_2 \cos^2\theta + ... + a_N \cos^N\theta \quad (7)$$

where we explicitly add the subscript $\mathbf{a}$ to denote the vector of coefficients $\mathbf{a} = [a_0, a_1, ..., a_N] \in \mathbb{R}^{N+1}$. Since any normalization of (7) would not affect the directional characteristic of the microphone, it is convenient to set $a_0 = 1 - a_1 - a_2 - ... - a_N$, such that $\Gamma_{\mathbf{a}}(0) = 1$. It should also be observed that $\Gamma_{\mathbf{a}}(\theta)$ is an even function of θ. Therefore the design problem can be restricted to the angular sector $[0, \pi]$ without loss of generality.

The most widely used directivity patterns are *subcardioid*, *cardioid*, *hypercardioid* and *supercardioid*. The directivity pattern of the N^{th}-order cardioid microphone is $\Gamma(\theta) = (0.5 + 0.5\cos\theta)^N$ [6]. Subcardioid microphones, which are frequently used in classical recordings, are defined only as first-order microphones, and have $\mathbf{a} = [0.7, 0.3]$. Unlike cardioid and subcardioid, other directivity patterns, such as supercardioid and hypercardioid, are designed to satisfy specific criteria. Supercardioid is the pattern with the maximum *front-back ratio* [6], which is defined as:

$$F(\mathbf{a}) = \frac{\int_0^{\pi/2} |\Gamma_{\mathbf{a}}(\theta)|^2 \sin\theta\, d\theta}{\int_{\pi/2}^{\pi} |\Gamma_{\mathbf{a}}(\theta)|^2 \sin\theta\, d\theta}, \quad (8)$$

under the assumption of spherically isotropic noise and axisymmetric directivity pattern. The coefficients $a_1, ..., a_N$ that maximize (8) are already known, and can be found in [6].

Hypercardioid, is the microphone with the maximum *directivity factor*. The directivity factor, under the assumption of spherically isotropic noise and axisymmetric directivity pattern, is defined [6] as:

$$Q(\mathbf{a}) = \frac{|\Gamma_{\mathbf{a}}(0)|^2}{\frac{1}{\pi}\int_0^{\pi} |\Gamma_{\mathbf{a}}(\theta)|^2 \sin\theta\, d\theta}. \quad (9)$$

As with the case of the supercardioid pattern, the coefficients which maximize (9) are known [6].

3. GENERALIZED DIRECTIVITY PATTERN DESIGN METHOD

As discussed in Sec. 2, the coefficients $\mathbf{a}$ are already known for the mentioned optimal criteria - maximum front/back ratio, and maximum directivity factor. However, there are applications in which the sources of interest are located in an angular sector different from π or in a narrow sector in front of the microphone. Furthermore, a more uniform directivity throughout the angular sector of interest would be preferable, such that the sources of interest are recorded with the same intensity. In this section we formulate a cost function whose minima provides directivity patterns which satisfy these two criteria. It is then shown that popular directivity patterns are also particular solutions of the same optimization problem.

3.1. Optimization problem definition

We propose selecting the coefficients $\mathbf{a}$ as the solution of the minimization problem:

$$\tilde{\mathbf{a}}(\alpha, \lambda) = \underset{\mathbf{a}}{\operatorname{argmin}}\ \Phi_{\mathbf{a}}(\alpha, \lambda) \quad \text{subject to } \Gamma_{\mathbf{a}}(\theta) \leq 1 \forall\theta, \quad (10)$$

where the cost function is given by:

$$\Phi_{\mathbf{a}}(\alpha, \lambda) = \lambda \frac{\int_{\alpha}^{\pi} |\Gamma_{\mathbf{a}}(\theta)|^2 \sin\theta\, d\theta}{\int_0^{\alpha} |\Gamma_{\mathbf{a}}(\theta)|^2 \sin\theta\, d\theta} + (1-\lambda) \int_0^{\alpha} |\Gamma'_{\mathbf{a}}(\theta)|^2 \sin\theta\, d\theta, \quad (11)$$

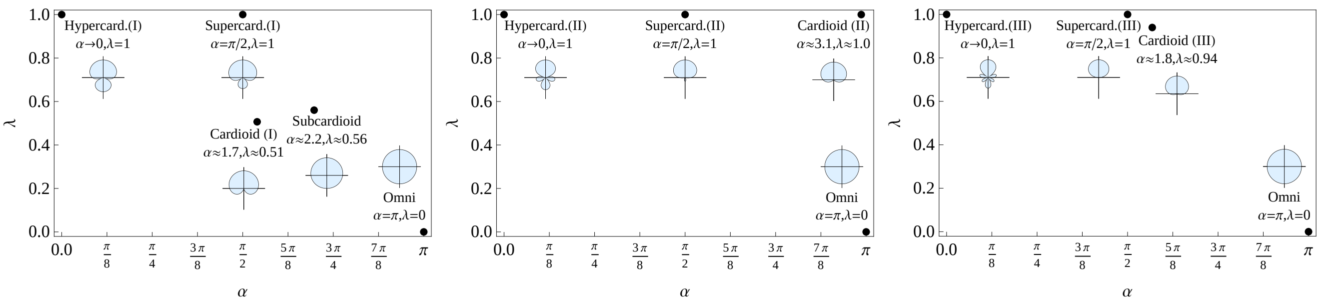
with $\lambda \in [0, 1]$ and $\alpha \in [0, \pi]$. This cost function is a convex combination of two terms; (a) the ratio between the directional gains in the frontal sector $[0, \alpha]$ and in the complementary sector $[\alpha, \pi]$, and (b) a term related to the smoothness of the directivity pattern in the angular sector $[0, \alpha]$. The optimization problem defined in this manner generates a family of directivity patterns, each corresponding to a fixed value of (α, λ).

The cost function is designed in this manner so that it allows for explicit control of two parameters which are most important for many recording applications. The angle α can be set to cover the angular region where the sound sources of interest are located, while λ controls the relative importance of the uniformity of the directivity in the desired region (such that the sources of interest are recorded at the same level) and the suppression of sources outside of it. From this point of view, the proposed method provides a convenient interface to microphone adjustment, in the sense that it directly sets the two immediately relevant parameters α and λ, rather than adjusting the N coefficients in an ad-hoc manner without a clear impact on the shape of the directivity.

Other choices of the cost function could have been made. For instance, (11) could be replaced by:

$$\lambda \int_{\alpha}^{\pi} |\Gamma_{\mathbf{a}}(\theta)|^2 \sin\theta\, d\theta + (1-\lambda) \int_0^{\alpha} |1 - \Gamma_{\mathbf{a}}(\theta)|^2 \sin\theta\, d\theta, \quad (12)$$

where the two terms reflect the leakage of energy from outside of the region of interest and the uniformity of the directivity pattern in



(a) First-order (b) Second-order (c) Third-order

Fig. 1. Overview of the design space with markers on the standard directivity patterns for microphone orders $N = 1, 2, 3$.

the desired range, respectively. However, we observed that solutions of such an optimization problem usually exhibit undesired ripples. Furthermore, the standard directivity patterns did not fit in the optimization framework, while on the other hand they are particular solutions of the optimization problem defined in (10),(11).

3.2. Relation to standard directivity patterns

Standard directivity patterns, and their relation to the cost function proposed in this paper are discussed below.

a) Supercardioid. When $\lambda = 1$ and $\alpha = \frac{\pi}{2}$, the cost function (11) becomes:

$$\Phi_{\mathbf{a}}\left(\frac{\pi}{2}, 1\right) = \frac{\int_{\frac{\pi}{2}}^{\pi} |\Gamma_{\mathbf{a}}(\theta)|^2 \sin\theta \, d\theta}{\int_0^{\frac{\pi}{2}} |\Gamma_{\mathbf{a}}(\theta)|^2 \sin\theta \, d\theta} \tag{13}$$

that is the inverse of the front-back ratio (8) for axisymmetric directivity patterns and under the assumption of spherically isotropic noise[1]. As a consequence, the solution of the associated minimization problem (10) is the supercardioid pattern.

b) Hypercardioid. When $\lambda = 1$ and $\alpha \to 0$, the optimization problem reduces to:

$$\tilde{\mathbf{a}}\,(0, 1) = \underset{\mathbf{a}}{\operatorname{argmin}} \int_0^{\pi} |\Gamma_{\mathbf{a}}(\theta)|^2 \sin\theta \, d\theta \tag{15}$$

whose solution is the hypercardioid pattern. In fact, given the constraint $\Gamma_{\mathbf{a}}(0) = 1$, solving the problem (15) is equivalent to maximizing the directivity factor (9).

c) Omnidirectional. When $\lambda = 0$ and $\alpha = \pi$, the cost function becomes:

$$\Phi_{\mathbf{a}}(\pi, 0) = \int_0^{\pi} |\Gamma'_{\mathbf{a}}(\theta)|^2 \sin\theta \, d\theta. \tag{16}$$

Since $|\Gamma'_{\mathbf{a}}(\theta)|^2$ and $\sin\theta$ are non-negative functions of $\theta \in [0, \pi]$, the solution of the associated minimization problem is the constant function $\Gamma(\theta) = 1$, i.e. the omnidirectional pattern.

d) Subcardioid and cardioid. These directivity patterns were not originally designed to satisfy any specific optimal criteria, and are not particular cases of the cost function (11). However, it is shown below that they are very close to solutions of the optimization problem.

[1]This paper is focused on the case of spherically isotropic noise. The case of cylindrically isotropic noise can be studied by redefining the cost function (11) as

$$\Phi_{\mathbf{a}}(\alpha, \lambda) = \lambda \frac{\int_{\alpha}^{\pi} |\Gamma_{\mathbf{a}}(\theta)|^2 \, d\theta}{\int_0^{\alpha} |\Gamma_{\mathbf{a}}(\theta)|^2 \, d\theta} + (1 - \lambda) \int_0^{\alpha} |\Gamma'_{\mathbf{a}}(\theta)|^2 \, d\theta. \tag{14}$$

Table 1. Relation to standard directivity patterns.

Pattern	N	α	λ	Error δ [dB]
Omnidirectional	0	π	0	$-\infty$
Hypercardioid	Any	0	1	$-\infty$
Supercardioid	Any	$\frac{\pi}{2}$	1	$-\infty$
Subcardioid	1	2.190	0.5602	$\ll -100$
Cardioid (I)	1	1.696	0.5066	$\ll -100$
Cardioid (II)	2	3.099	1.000	-87
Cardioid (III)	3	1.785	0.9399	-52
Cardioid (IV)	4	2.464	1.000	-69

In order to measure the distance between a desired directivity pattern $\Gamma_d(\theta)$ (e.g. N^{th}-order cardioid or subcardioid) and the space of solutions generated by (10),(11), the following approach is used: among all the directivity functions, $\Gamma_{\tilde{\mathbf{a}}(\alpha,\lambda)}$, that are solutions of the problem (10),(11), the closest to $\Gamma_d(\theta)$ in the least square sense is found, and the average residual is used as a distance metric:

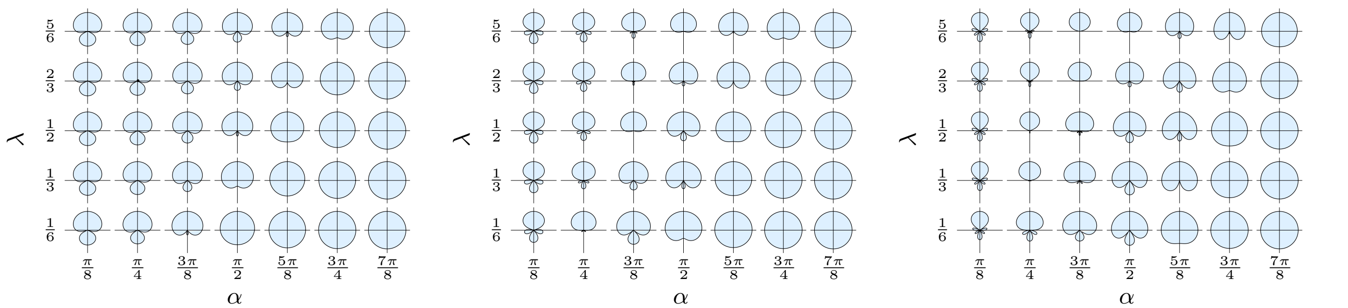
$$\delta = \min_{\alpha,\lambda} \frac{1}{2\pi} \int_0^{2\pi} \left|\Gamma_{\tilde{\mathbf{a}}(\alpha,\lambda)}(\theta) - \Gamma_d(\theta)\right|^2 d\theta\,. \tag{17}$$

Table 1 shows errors of closest solutions, obtained using a grid search, for subcardioid and first to fourth order cardioid microphones. For the first-order patterns, the approximation error $\Delta = 10\log_{10}\delta$ is less than -100dB. Moreover, while their values of λ are similar, the value of α for subcardioid is larger than that of cardioid, supporting the fact that subcardioid microphones have a wider aperture with flat response. Higher-order cardioid patterns, all have errors smaller than -50dB which is negligible for all practical purposes, thus indicating that they fit in the proposed optimization problem.

The results presented so far are summarized for $N = 1, 2, 3$ in Fig 1. Every point in these figures is representative of an (α, λ) pair, and its associated optimal directivity pattern $\Gamma_{\tilde{\mathbf{a}}(\alpha,\lambda)}$.

3.3. Design examples

The optimization framework described in (10) and (11) generates a much richer class of solutions than the standard directivity patterns discussed earlier. Some of these solutions are shown in Fig. 2. These solutions were obtained by means of the Nelder-Mead optimization



(a) First-order (b) Second-order (c) Third-order

Fig. 2. Examples of directivity patterns generated by the design framework for different (α, λ) pairs, and for microphone orders $N = 1, 2, 3$. The figures are plotted in dB scale.

algorithm, which was found to be reliable and very fast for the purpose of this application. However, any other constrained optimization algorithms can also be used[2].

Several directivity patterns of practical use can be obtained by means of the proposed design method. For instance, small values of α produce directivity patterns similar to hypercardioid, but with a slightly larger frontal beam. These patterns can be used to record more than one instrument (or more than one actor, in the case of the film industry) from distance. As a second example, there may be a need to record all sound sources in the frontal plane (e.g. orchestra recording). The working line on the (α, λ) plane, is $\alpha = \frac{\pi}{2}$ in this case. At the top end ($\lambda = 1$), the supercardioid microphone is the one that maximizes the front-back ratio. However, the supercardioid pattern causes sound sources near $\pi/2$ to be highly attenuated. This drawback can be alleviated by selecting smaller values of λ, thus trading a smaller front-back ratio with a smoother directivity function in the frontal plane.

Directivity patterns that reject sound sources in a given angular range at the back of the microphone (i.e. for $\theta \in [\pi-\epsilon, \pi+\epsilon]$), could also be of interest. Two practical uses could be to cancel sound from a noise source with known position, or to reduce acoustic feedback in acoustic reinforcement systems. To this end, the easiest solution for an N-order microphone would be to impose the polynomial (7) to have a zero of multiplicity N at $\theta = \pi$. However, in this way, the directivity would abruptly go to zero in a region around $\theta = \pi$, without control on the width of such a region. A better control would be available if the proposed method is employed. This kind of directivity patterns is realized for $\alpha \to \pi$, and values of λ close to unity, i.e. the points on the (α, λ) plane near the top right corner.

The proposed method can also be used for applications where an automatic selection of the directivity pattern is needed. For instance, in a teleconferencing scenario, the directivity pattern could be widened as the estimation of the position of the speaker becomes less accurate. As a similar case, the built-in microphone of a video camera could be coupled with the optical lens, so as to provide an "acoustical zoom" that matches the optical zoom. To this end, the operating line between the two points $(0, 1)$ and $(\pi, 0)$ can be used, where the patterns evolve from highly directional to omnidirectional.

[2]A Mathematica notebook which demonstrates the generalized design approach proposed in this paper is made available at http://www.enzo.desena.name/gddp . The software also provides guidance for selecting λ other than by the visual inspection of the directivity pattern plot. Possible approaches are setting λ as to obtain (i) a certain attenuation at a given angle, (ii) a certain front-back ratio, or (ii) a certain directivity index.

4. CONCLUSIONS

A generalized design method for directivity patterns of spherical microphone arrays was introduced in this paper. The proposed method yields the coefficients of the microphone directivity pattern that minimize a cost function that has two free parameters, α and λ, which have an application-oriented meaning, thus providing a simple interface for sound engineers and technicians. It was found that most of the standard directivity patterns are solutions of this minimization problem with particular pairs of λ and α. As the two parameters can be freely varied, more general solutions can be obtained. Several examples of practical use were identified.

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